

Restoring broken entanglement by injecting separable correlations

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The distribution of entanglement is central in many protocols of quantum information and computation. However it is also known to be a very fragile process when loss and noise come into play. The inevitable interaction of the quantum systems with the external environment induces effects of decoherence which may be so strong to destroy any input entanglement, a phenomenon known as “entanglement breaking”. Here we study this catastrophic process in a correlated-noise environment showing how the presence of classical-type correlations can restore the distribution of entanglement. In particular, we consider a Gaussian environment whose thermal noise is strong enough to break the entanglement of two bosonic modes. In this scenario, we show that the injection of separable correlations from the same environment is able to reactivate the broken entanglement. This paradoxical effect happens both in schemes of direct distribution, where a third party (Charlie) broadcasts entangled states to remote parties (Alice and Bob), and in schemes of indirect distribution which are based on the protocol of entanglement swapping, whose theory is here generalized. Furthermore, the amount of entanglement activated by the injection can be large enough to be distilled using one-way distillation protocols. As a result, entanglement distribution and its distillation are still possible in the presence of entanglement-breaking channels, as long as a sufficient amount of separable correlations can be identified in the environment. These findings pose fundamental questions about the intimate interplay between local and nonlocal correlations, and suggest new perspectives for quantum repeaters in the presence of memory channels and correlated-noise environments, such as those characterized by non-Markovian dynamics.

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I. INTRODUCTION

Entanglement is a fundamental physical resource in quantum information and computation [1, 2]. Once that two remote parties, say Alice and Bob, share a suitable amount of entanglement, they can implement a variety of powerful protocols, including the teleportation of quantum states [3, 4] and quantum gates [5], as well as the distribution of unconditionally secure keys [6, 7]. The problem of entanglement distribution is therefore a central topic of investigation in the quantum information community. Unfortunately, this important resource is also fragile: Quantum systems inevitably interact with the external environment whose action may degrade or even destroy the presence of entanglement. Thus, in realistic implementations where the effect of decoherence is non-negligible, the distribution of entanglement becomes challenging and may need the help of distillation protocols [8, 9], where a large number of weakly-entangled states are transformed into a smaller number of strongly-entangled states via local operations and classical communications (LOCCs).

Decoherence affects all the possible forms of entanglement distribution. In a scheme of direct distribution, there is a middle station (Charlie) possessing a bipartite system in an entangled state; one subsystem is sent to Alice and the other to Bob. Alternatively, we may consider a scheme of indirect distribution, known as entanglement swapping [10–12], where the distribution is mediated by a measurement process. Here Alice and Bob each has a bipartite system prepared in an entangled state. One subsystem is retained while the other is sent to Charlie.

At his station, Charlie detects the two incoming subsystems by performing a suitable measurement (known as the “Bell measurement”) and communicates the classical outcome back to Alice and Bob. As a result of this process, the two subsystems retained by the remote parties are projected into an entangled state.

In both the configurations, the distribution and distillation of entanglement is possible as long as the decoherent action of the environment is not strong. When decoherence is strong enough to destroy any input entanglement, the environment is described by the notion of entanglement breaking (EB) channel [13, 14]. A quantum channel \mathcal{E} is EB when its local action on one part of a bipartite state always results into a separable output state. In other words, given two systems, A and B , in an arbitrary bipartite state ρ_{AB} , the output state $\rho_{AB'} = (\mathcal{I}_A \otimes \mathcal{E}_B)(\rho_{AB})$ is always separable, where \mathcal{I}_A is the identity channel applied to system A and \mathcal{E}_B is the EB channel applied to system B . Thus, if the input systems A and B were initially entangled (denoted by the notation $A - B$), the output systems A and B' are separable (denoted by the notation $A|B'$).

Despite EB channels have been the subject of an intensive study by the community, they have been only analyzed under Markovian conditions of no memory for the environment. In other words, when the distribution involves two or more systems travelling in a completely decohering environment, these systems are typically assumed to be perturbed in an independent fashion, each of them subject to the same memoryless channel.

For instance, consider the case of direct distribution depicted in Fig. 1(1). In the standard memoryless de-

scription of the environment, the entangled state ρ_{AB} of the input systems A and B is subject to a tensor product of channels $\mathcal{E}_A \otimes \mathcal{E}_B$. In this case, there is no way to distribute entanglement among any of the parties if both \mathcal{E}_A and \mathcal{E}_B are EB channels. Suppose that Charlie tries to share entanglement with one of the remote parties by sending one of the two systems while keeping the other (one-system transmission). For instance, Charlie may keep system A while transmitting system B to Bob. The action of $\mathcal{I}_A \otimes \mathcal{E}_B$ destroys the initial entanglement, so that systems A (kept) and B' (transmitted) are separable ($A|B'$). Symmetrically, the action of $\mathcal{E}_A \otimes \mathcal{I}_B$ destroys the entanglement between system A' (transmitted) and system B (kept), i.e., we have $A'|B$.

Now suppose that Charlie sends both his systems to Alice and Bob (two-system transmission). This strategy must necessarily fail under the assumption of no correlations in the environment. Since the joint action of the two EB channels is given by the tensor product $\mathcal{E}_A \otimes \mathcal{E}_B = (\mathcal{E}_A \otimes \mathcal{I}_B)(\mathcal{I}_A \otimes \mathcal{E}_B)$ quantum entanglement must necessarily be destroyed. In other words, since we have one-system EB ($A|B'$ and $A'|B$) then we must have two-system EB ($A'|B'$).

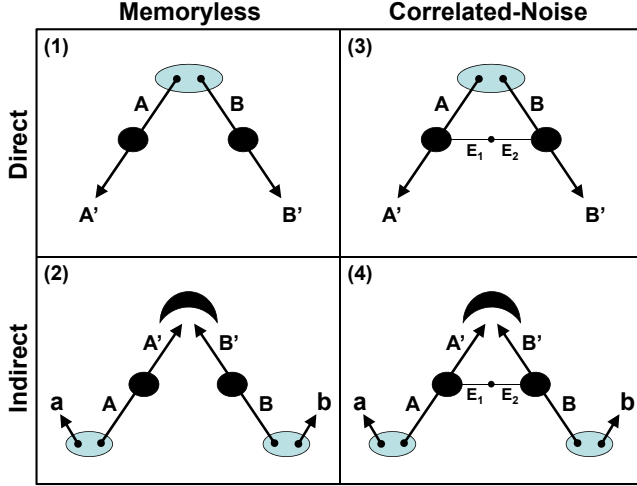


FIG. 1: Direct and indirect schemes for entanglement distribution in memoryless and correlated-noise environments. Charlie is the middle station transmitting to or receiving systems from Alice (left station) and Bob (right station). Ellipses represent entangled states, black-circles represent channels, and detectors are Bell measurements. (1) Direct distribution in a memoryless environment. If the individual channels are EB, then no distribution of entanglement is possible. We have $A|B'$ and $A'|B$ which implies $A'|B'$. (2) Entanglement swapping in a memoryless environment. We have that $a|A'$ and $B'|b$ implies $a|b$. (3) Direct distribution in a correlated environment. Despite $A|B'$ and $A'|B$ we have that $A' - B'$ is possible. Furthermore, this can be realized by a separable environment. (4) Entanglement swapping in a correlated environment. Despite $a|A'$ and $B'|b$ we have that $a - b$ is possible. Furthermore, this is realizable by a separable environment.

The previous reasoning can be extended to the case of

indirect distribution as shown in Fig. 1(2). Since the environment is memoryless ($\mathcal{E}_A \otimes \mathcal{E}_B$), we have that the absence of entanglement before the Bell measurement ($a|A'$ and $B'|b$) is a sufficient condition for the swapping protocol to fail, i.e., the remote systems a and b remains separable ($a|b$). Similarly to the previous case, if one-system transmission does not distribute entanglement, then two-system transmission cannot lead to entanglement generation via the swapping protocol.

In our paper, we show that the previous implications for direct and indirect distribution of entanglement are false in the presence of a correlated-noise environment, i.e., two-system transmission can successfully distribute entanglement despite one-system transmission is subject to EB. To the best of our knowledge, this is the first work to study the process of EB in the presence of correlated-noise environments, proving the advantages of the EB channels with memory.

In our study, we will show that there are typical physical conditions under which the environmental correlations are able to reactivate the distribution of entanglement, therefore “breaking entanglement-breaking”. The most remarkable finding is that we do not need to consider an entangled environment: The injection of separable correlations from the environment is able to restore the distribution of entanglement.

To better clarify these points, consider the schemes of direct and indirect distribution in the presence of a correlated-noise environment which are depicted in Figs. 1(3) and (4). In the scheme direct distribution of Fig. 1(3), an input entangled state ρ_{AB} is jointly transformed into an output state $\rho_{A'B'} = \mathcal{E}_{AB}(\rho_{AB})$. We assume that the dilation of the composite channel \mathcal{E}_{AB} is realized by introducing a two-system environment, E_1 and E_2 , in a bipartite state $\rho_{E_1 E_2}$, which interacts with the systems via two identical unitaries U_{AE_1} (transforming A and E_1) and U_{BE_2} (transforming B and E_2). In other words, we have

$$\rho_{A'B'} = \text{Tr}_{E_1 E_2}[(U_{AE_1} \otimes U_{BE_2}) \times (\rho_{AB} \otimes \rho_{E_1 E_2})(U_{AE_1}^\dagger \otimes U_{BE_2}^\dagger)]. \quad (1)$$

If the environmental state is not tensor product, i.e., $\rho_{E_1 E_2} \neq \rho_{E_1} \otimes \rho_{E_2}$, then the composite channel cannot be decomposed into memoryless channels, i.e., $\mathcal{E}_{AB} \neq \mathcal{E}_A \otimes \mathcal{E}_B$. From the dilation given in Eq. (1), we can always define the reduced channels, \mathcal{E}_A and \mathcal{E}_B , acting on the individual systems. For instance, if only system B is transmitted, then we have the evolved state

$$\begin{aligned} \rho_{AB'} &= (\mathcal{I}_A \otimes \mathcal{E}_B)(\rho_{AB}) \\ &= \text{Tr}_{E_2}[(I_A \otimes U_{BE_2})\rho_{AB} \otimes \rho_{E_2}(I_A \otimes U_{BE_2}^\dagger)], \quad (2) \end{aligned}$$

where $\rho_{E_2} = \text{Tr}_{E_1}(\rho_{E_1 E_2})$. A similar formula holds for the evolution of the first system $\rho_{A'B} = (\mathcal{E}_A \otimes \mathcal{I}_B)(\rho_{AB})$.

In our work we show that, despite $\mathcal{I}_A \otimes \mathcal{E}_B$ and $\mathcal{E}_A \otimes \mathcal{I}_B$ are EB channels (so that $A|B'$ and $B|A'$), the composite channel \mathcal{E}_{AB} is able to preserve entanglement (so that

$A' - B'$ is possible). In other words, we show a paradoxical situation where Charlie is not able to share entanglement with Alice or Bob, but still he can distribute entanglement to them. This is clearly an effect of the injected correlations coming from the environmental state $\rho_{E_1 E_2}$. As mentioned earlier, our main finding is that these correlations do not need to be strong: Entanglement distribution can be activated by separable correlations, i.e., by an environment which is in a separable state $\rho_{E_1 E_2}$.

This paradoxical result can also be extended to entanglement distillation, which typically requires stronger conditions than entanglement distribution (demanded by the existence of effective distillation protocols). Despite the individual channels are EB, their combination into a separable environment enables Charlie to distribute distillable entanglement to Alice and Bob. This is easy to prove for an environment with finite memory, which can be decomposed as $\mathcal{E}_{AB} \otimes \mathcal{E}_{AB} \otimes \dots$.

In our investigation, we also consider the case of entanglement swapping in a correlated-noise environment as depicted in Fig. 1(4). Here Alice and Bob have two entangled states, ρ_{aA} and ρ_{Bb} , respectively. Systems a and b are retained, while systems A and B are transmitted to Charlie, therefore undergoing the joint quantum channel \mathcal{E}_{AB} . Before the Bell measurement, the global state is described by

$$\begin{aligned} \rho_{aA'B'b} &= (\mathcal{I}_a \otimes \mathcal{E}_{AB} \otimes \mathcal{I}_b)(\rho_{aA} \otimes \rho_{Bb}) \\ &= \text{Tr}_{E_1 E_2}[U(\rho_{aA} \otimes \rho_{E_1 E_2} \otimes \rho_{Bb})U^\dagger], \end{aligned} \quad (3)$$

where $U = I_a \otimes U_{AE_1} \otimes U_{E_2 B} \otimes I_b$. As before, we consider the case where the reduced channels, \mathcal{E}_A and \mathcal{E}_B , are EB channels, so that no entanglement survives before the Bell measurement ($a|A'$ and $B'|b$). If the environment has no memory ($\rho_{E_1 E_2} = \rho_{E_1} \otimes \rho_{E_2}$) there is no way to distribute entanglement to Alice and Bob ($a|b$). By contrast, if the environment has memory ($\rho_{E_1 E_2} \neq \rho_{E_1} \otimes \rho_{E_2}$), then entanglement distribution is possible ($a - b$) and this distribution can be activated by a separable environmental state $\rho_{E_1 E_2}$. Thus, we have the paradoxical situation where no bipartite entanglement survives at Charlie's station ($a|A'$ and $B'|b$), but still the swapping protocol is able to generate remote entanglement at Alice's and Bob's stations ($a - b$) thanks to the separable correlations injected by the environment.

As before, these separable correlations can be strong enough to distribute distillable entanglement to the remote parties. Note that this suggests a preferred mechanism for quantum repeaters [15] in the presence of correlated-noise environments, where Charlie first performs the swapping protocol on each pair of systems received from the remote parties, followed by Alice and Bob performing entanglement distillation on the whole collection of swapped states. While this mechanism can be successful, the other standard approach of distillation followed by swapping cannot work since Charlie is not able to distill any entanglement with Alice or Bob.

Our results are derived for continuous-variable systems, i.e., quantum systems with an infinite-dimensional Hilbert space [16–18]. In particular, we have considered the bosonic modes of the electromagnetic field (so that the various systems depicted in Fig. 1 can be interpreted as bosonic modes.) The input modes are prepared in Gaussian states with Einstein-Podolsky-Rosen (EPR) correlations [18, 19], which are the most typical form of continuous variable entanglement, and they are assumed to evolve under the action of a Gaussian environment. This type of environment is modelled by two beam splitters which mix the travelling modes, A and B , with two environmental modes, E_1 and E_2 , prepared in a bipartite Gaussian state $\rho_{E_1 E_2}$ (separable or entangled). The reduced channels, \mathcal{E}_A and \mathcal{E}_B , are two lossy channels whose transmissivities and thermal noises are such to make them EB channels. The adoption of bosonic systems and Gaussian channels is interesting not only for the potential practical applications, but also because the Gaussian formalism enables us to find closed analytical results.

In terms of potential impact, our work opens new perspectives for entanglement distribution and distillation in correlated-noise environments and memory channels, where the presence of correlations can be exploited to recover from the breaking of entanglement. Since the entanglement restoration can be achieved by the injection of separable correlations, our work poses fundamental questions on the intimate relations and joint dynamics of local and nonlocal correlations. It is important to note that memory channels and correlated (in particular, non-Markovian) environments are present in a wide series of practical scenarios, including spin chains [20], atoms in optical cavities [21], quantum dots in photonic crystals [22], photonic propagation through linear optical systems [23–25] and atmospheric turbulence [26–28].

The paper is structured as follows. In Sec. II we provide basic notions on bosonic systems, Gaussian states and their main properties, which are useful for understanding the remainder of the paper. (This section may be skipped by those readers who are familiar with continuous variable quantum information). In Sec. III we introduce and characterize the basic model of correlated Gaussian environment, which directly generalizes the standard model of thermal-loss environment. We identify the physical conditions under which the correlated Gaussian environment is separable or entangled, discussing specific cases where the quadrature correlations are symmetric or asymmetric. In Sec. IV, we study the direct distribution of entanglement in the presence of the correlated Gaussian environment and assuming the condition of one-system EB. We provide the regimes of parameters under which remote entanglement is activated by environmental correlations (in particular, separable correlations) and the stronger regimes where the generated remote entanglement is also distillable. The nature of the environmental correlations is also discussed in terms of quantum discord and the specific dynamics of the EPR

correlations is analyzed. In Sec. V, we generalize the theory of entanglement swapping to the correlated Gaussian environment (with more details provided in appendix A). We first consider the swapping of EPR correlations and then the swapping and distillation of entanglement, finding the regimes of parameters where swapping and distillation are successful despite the EB condition. Potential implications for quantum repeaters are also discussed. Finally, Sec. VI is for conclusion and discussion.

II. BASIC NOTIONS ON BOSONIC SYSTEMS AND GAUSSIAN STATES

The most important systems in continuous variable quantum information are the bosonic modes of the electromagnetic field. A bosonic mode is a quantum system with an infinite-dimensional Hilbert space and described by a pair of quadrature operators: Position \hat{q} and momentum \hat{p} , satisfying the commutation relation $[\hat{q}, \hat{p}] = 2i$. More generally, a bosonic system of n modes is described by a vector of $2n$ quadrature operators

$$\hat{\mathbf{x}}^T := (\hat{q}_1, \hat{p}_1, \dots, \hat{q}_n, \hat{p}_n), \quad (4)$$

satisfying $[\hat{x}_i, \hat{x}_j] = 2i\Omega_{ij}$, where $i, j = 1, \dots, 2n$ and Ω_{ij} is the generic element of the symplectic form

$$\Omega := \bigoplus_{k=1}^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (5)$$

In experimental quantum optics, bosonic modes are typically prepared in Gaussian states. By definition, a quantum state ρ is ‘‘Gaussian’’ if its Wigner phase-space representation is Gaussian [18]. This feature implies that any measurement of the quadratures provides outcomes which are distributed according to a Gaussian statistics. From a theoretical point of view, a Gaussian state is very easy to characterize, being completely characterized by its first and second-order statistical moments.

The first-order moment is known as the mean value, and is defined by $\bar{\mathbf{x}} := \langle \hat{\mathbf{x}} \rangle$, where $\langle \hat{O} \rangle := \text{Tr}(\hat{O}\rho)$ denotes the average of the arbitrary operator \hat{O} on the state ρ . The second-order moment is known as the covariance matrix (CM) \mathbf{V} , with generic element

$$V_{ij} := \frac{1}{2} \langle \{\Delta\hat{x}_i, \Delta\hat{x}_j\} \rangle, \quad (6)$$

where $\Delta\hat{x}_i := \hat{x}_i - \bar{x}_i$ is the deviation and $\{, \}$ is the anti-commutator. Note the diagonal elements V_{ii} correspond to the variances of the quadratures

$$V(\hat{x}_i) := \langle \Delta\hat{x}_i^2 \rangle = \langle \hat{x}_i^2 \rangle - \bar{x}_i^2. \quad (7)$$

The CM is a $2n \times 2n$ real symmetric matrix, which must satisfy the uncertainty principle [29]

$$\mathbf{V} + i\Omega \geq 0, \quad (8)$$

implying the positive-definiteness $\mathbf{V} > 0$ [30].

The simplest examples of Gaussian states are thermal states. A single-mode thermal state has zero mean ($\bar{\mathbf{x}} = \mathbf{0}$) and CM equal to $\mathbf{V} = (2\bar{n} + 1)\mathbf{I}$, where \mathbf{I} is the 2×2 identity matrix and \bar{n} is the average number of thermal photons. This state collapses to the vacuum state for $\bar{n} = 0$. Multimode thermal states can be easily constructed by tensor product. Remember that the operation of tensor product for the states $\rho_1 \otimes \rho_2$ corresponds to the operation of direct sum for the CMs $\mathbf{V}_1 \oplus \mathbf{V}_2$. Then, the opposite operation of partial trace $\rho_1 = \text{Tr}_2(\rho_{12})$ corresponds to collapsing the global CM \mathbf{V}_{12} into the diagonal block associated with (\hat{q}_1, \hat{p}_1) .

In quantum optics’ labs, Gaussian states are typically processed by Gaussian operations. The simplest ones are Gaussian unitaries, defined as those unitary operators which transform Gaussian states into Gaussian states. At the level of the statistical moments, the action of a Gaussian unitary $\rho \rightarrow U\rho U^\dagger$ corresponds to

$$\bar{\mathbf{x}} \rightarrow \mathbf{S}\bar{\mathbf{x}} + \mathbf{d}, \quad \mathbf{V} \rightarrow \mathbf{S}\mathbf{V}\mathbf{S}^T, \quad (9)$$

where \mathbf{d} is a real displacement vector and \mathbf{S} is a symplectic matrix, i.e., a real matrix preserving the symplectic form $\mathbf{S}\Omega\mathbf{S}^T = \Omega$. In the Heisenberg picture, a Gaussian unitary corresponds to the affine map

$$\hat{\mathbf{x}} \rightarrow \mathbf{S}\hat{\mathbf{x}} + \mathbf{d}. \quad (10)$$

A Gaussian unitary is said to be a canonical unitary when $\mathbf{d} = \mathbf{0}$. The most important example of canonical unitary is the beam-splitter transformation, which involves two bosonic modes. This unitary is characterized by a transmissivity parameter $0 \leq \tau \leq 1$ and corresponds to the symplectic matrix

$$\mathbf{S}(\tau) = \begin{pmatrix} \sqrt{\tau}\mathbf{I} & \sqrt{1-\tau}\mathbf{I} \\ -\sqrt{1-\tau}\mathbf{I} & \sqrt{\tau}\mathbf{I} \end{pmatrix}. \quad (11)$$

Other important examples of Gaussian operations are the Gaussian channels [31]. These channels describe all those cases where bosonic systems and environment interact by means of linear and/or bilinear Hamiltonians. The action of a Gaussian channel $\rho \rightarrow \mathcal{E}(\rho)$ corresponds to the following transformation for the CM of the state

$$\mathbf{V} \rightarrow \mathbf{K}\mathbf{V}\mathbf{K}^T + \mathbf{N}, \quad (12)$$

where \mathbf{K} and $\mathbf{N} = \mathbf{N}^T$ are $2n \times 2n$ real matrices, satisfying the condition $\mathbf{N} + i\Omega - i\mathbf{K}\Omega\mathbf{K}^T \geq 0$ [32].

The most important example of Gaussian channel is the lossy channel, which is typically used to model the optical propagation through dissipative linear media. When a single bosonic mode propagates through a lossy channel, its CM is transformed by Eq. (12) with

$$\mathbf{K} = \sqrt{\tau}\mathbf{I}, \quad \mathbf{N} = (1 - \tau)(2\bar{n} + 1)\mathbf{I}, \quad (13)$$

where $0 \leq \tau \leq 1$ is the transmissivity of the channel and $\bar{n} \geq 0$ its thermal number. A single-mode lossy channel can be dilated into a beam-splitter with transmissivity τ , which mixes the incoming mode with an environmental thermal mode containing \bar{n} mean photons.

A. Symplectic spectrum

A central result in the theory of Gaussian states is Williamson's theorem [18, 33]. Given a CM \mathbf{V} , there always exists a symplectic matrix \mathbf{S} realizing the diagonalization

$$\mathbf{V} = \mathbf{S} \mathbf{W} \mathbf{S}^T, \quad \mathbf{W} = \bigoplus_{k=1}^n \nu_k \mathbf{I}, \quad (14)$$

where the diagonal matrix \mathbf{W} is known as the Williamson normal form, and $\{\nu_1, \dots, \nu_n\}$ are the n symplectic eigenvalues of the CM. Since $\det \mathbf{S} = 1$, we have that

$$\det \mathbf{V} = \prod_{k=1}^n \nu_k^2. \quad (15)$$

Using the symplectic spectrum, we can write the uncertainty principle of Eq. (8) in the equivalent form

$$\mathbf{V} > 0, \quad \nu_k^2 \geq 1, \quad (16)$$

which clearly implies $\nu_k \geq 1$ [34]. The symplectic spectrum fully determines the entropic and purity properties of the Gaussian states. In fact, the von Neumann entropy $S(\rho) := -\text{Tr}(\rho \log \rho)$ of an arbitrary n -mode Gaussian state is expressed by [18]

$$S(\rho) = \sum_{k=1}^n h(\nu_k), \quad (17)$$

where

$$h(x) := \frac{x+1}{2} \log \frac{x+1}{2} - \frac{x-1}{2} \log \frac{x-1}{2}. \quad (18)$$

The basis of the logarithm can be taken to be 2 for bits or the Euler's number "e" for nats. The purity of a Gaussian state is given by

$$\mu(\rho) := \text{Tr} \rho^2 = \frac{1}{\sqrt{\det \mathbf{V}}} = \prod_{k=1}^n \nu_k^{-1}, \quad (19)$$

so that the state is pure (mixed) iff $\det \mathbf{V} = 1$ (> 1).

In our paper, we will consider situations where the symplectic eigenvalues are large. In this case, it is useful to use the asymptotic expansion

$$h(x) \simeq \log \frac{e}{2} x + O\left(\frac{1}{x}\right), \quad (20)$$

which is valid for large x . For instance, if the whole symplectic spectrum is diverging (ν_k large for any k), then we can use the expansion of Eq. (20) and write the following asymptotic formula

$$S(\rho) \simeq \log \left(\frac{e}{2}\right)^n \sqrt{\det \mathbf{V}} = \log \frac{1}{\mu(\rho)} + n \log \frac{e}{2}, \quad (21)$$

where the entropy is simply related to the CM and the purity of the Gaussian state.

B. Two-mode Gaussian states and bipartite entanglement

Since we will consider the distribution of bipartite entanglement, we devote a specific section to Gaussian states of two bosonic modes and their separability properties. Let us consider two modes, say A and B , with quadrature vector $\hat{\mathbf{x}}^T := (\hat{q}_A, \hat{p}_A, \hat{q}_B, \hat{p}_B)$. We assume that these modes are described by a zero-mean Gaussian state ρ_{AB} . Since $\bar{\mathbf{x}} = \mathbf{0}$, this state is fully characterized by its CM, that we can always express in the blockform

$$\mathbf{V} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{B} \end{pmatrix}, \quad (22)$$

where \mathbf{A} , \mathbf{B} and \mathbf{C} are 2×2 matrices. Finding the symplectic spectrum is straightforward, since [18, 36]

$$\nu_{\pm} = \sqrt{\frac{\Delta \pm \sqrt{\Delta^2 - 4 \det \mathbf{V}}}{2}}, \quad (23)$$

where $\Delta := \det \mathbf{A} + \det \mathbf{B} + 2 \det \mathbf{C}$. The uncertainty principle is then equivalent to the bona-fide condition

$$\mathbf{V} > 0, \quad \nu_{-}^2 \geq 1, \quad (24)$$

where ν_{-} is the smallest symplectic eigenvalue.

It is very easy to study the separability properties. It is sufficient to compute the smallest partially-transposed symplectic (PTS) eigenvalue ε . This eigenvalue can be computed using the formula of ν_{-} , given in Eq. (23), where we replace Δ with $\tilde{\Delta} = \det \mathbf{A} + \det \mathbf{B} - 2 \det \mathbf{C}$. Then, the Gaussian state is separable (entangled) if and only if $\varepsilon \geq 1$ ($\varepsilon < 1$). More strongly, this eigenvalue provides a quantification of entanglement, being an entanglement monotone for two-mode Gaussian states. In fact, it is monotonically related to the log-negativity [37]

$$\mathcal{E} = \max \{0, -\log \varepsilon\}, \quad (25)$$

which is itself an entanglement monotone.

The log-negativity \mathcal{E} also provides an upper-bound to the distillable entanglement, i.e., the average number of entanglement bits per copy which can be extracted from infinitely-many copies of the state $\rho_{AB} \otimes \rho_{AB} \otimes \dots$. More interestingly, we can consider a lower-bound to the distillable entanglement. According to the hashing inequality [38, 39], this is given by the coherent information [40, 41]

$$I(A)B = S(\rho_B) - S(\rho_{AB}), \quad (26)$$

where $\rho_B = \text{Tr}_A(\rho_{AB})$ is the reduced state of Bob, here corresponding to a zero-mean Gaussian state with CM \mathbf{B} . Using Eq. (17), the coherent information becomes

$$I(A)B = h(\nu_B) - h(\nu_{-}) - h(\nu_{+}), \quad (27)$$

where $\nu_B = \sqrt{\det \mathbf{B}}$ is the symplectic eigenvalue of \mathbf{B} , and $\{\nu_{-}, \nu_{+}\}$ is the symplectic spectrum of \mathbf{V} . In the

case where these eigenvalues are all very large, we can use the expansion of Eq. (20) to get the formula

$$I(A)B) \simeq \log \frac{2}{e} \sqrt{\frac{\det \mathbf{B}}{\det \mathbf{V}}} . \quad (28)$$

Remember that the coherent information is a lower bound to the distillable entanglement that can be obtained by means of one-way protocols between Alice and Bob. In these protocols, Alice applies a quantum instrument to her copies, i.e., a collection of completely-positive trace-preserving maps labelled by a classical index k . Then, she classically communicates k to Bob, who applies a conditional quantum operation on his copies (see Ref. [38] for more details).

C. EPR correlations

The most important example of two-mode Gaussian state is the two-mode squeezed vacuum state, also known as the EPR state. This state represents the most common source of continuous-variable entanglement, and will be adopted as such a source in our study of entanglement distribution.

An EPR state has zero mean and CM given by

$$\mathbf{V} = \begin{pmatrix} \mu \mathbf{I} & \mu' \mathbf{Z} \\ \mu' \mathbf{Z} & \mu \mathbf{I} \end{pmatrix} , \quad (29)$$

where $\mu \geq 1$, $\mu' := \sqrt{\mu^2 - 1}$, and

$$\mathbf{Z} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} . \quad (30)$$

The quadratures $\hat{\mathbf{x}}^T := (\hat{q}_A, \hat{p}_A, \hat{q}_B, \hat{p}_B)$ have equal variances μ and maximal correlations $\langle \hat{q}_A \hat{q}_B \rangle = -\langle \hat{p}_A \hat{p}_B \rangle = \mu'$. The parameter μ can be used to quantify the entanglement between the two modes A and B . In fact, the smallest PTS eigenvalue takes the form

$$\varepsilon = \sqrt{2\mu(\mu - \mu') - 1} , \quad (31)$$

which is monotonically decreasing in μ . We can easily check that we have entanglement ($\varepsilon < 1$) for any $\mu > 1$.

The EPR correlations can be characterized in terms of the variance of the EPR quadratures

$$\hat{q}_- := \frac{\hat{q}_A - \hat{q}_B}{\sqrt{2}} , \quad \hat{p}_+ = \frac{\hat{p}_A + \hat{p}_B}{\sqrt{2}} . \quad (32)$$

The EPR condition corresponds to $V(\hat{q}_-) = V(\hat{p}_+) < 1$, which means that the quadrature correlations between the two modes are below the vacuum noise. For the EPR state described by Eq. (29) indeed we have

$$V(\hat{q}_-) = V(\hat{p}_+) = \mu - \mu' , \quad (33)$$

which is less than 1 for any $\mu > 1$. For this type of state, EPR correlations and entanglement are equivalent conditions. In general, the presence of EPR correlations in an

arbitrary two-mode Gaussian state represents a sufficient condition for entanglement.

In the limit of large entanglement ($\mu \rightarrow \infty$), the EPR state becomes ideal. In fact, we have $V(\hat{q}_-) = V(\hat{p}_+) \rightarrow 0$, which corresponds to realizing the ideal EPR correlations $\hat{q}_A = \hat{q}_B$ and $\hat{p}_A = -\hat{p}_B$, where positions (momenta) are perfectly correlated (anticorrelated). Note that an alternative EPR state has $-\mathbf{Z}$ in the place of \mathbf{Z} in Eq. (29). This other state has EPR correlations in the quadratures \hat{q}_+ and \hat{p}_- , therefore tending to realize the ideal conditions $\hat{q}_A = -\hat{q}_B$ and $\hat{p}_A = \hat{p}_B$. In our work, we do not consider this alternative state, but similar results can easily be derived.

D. Gaussian quantum discord

Quantum entanglement is synonymous of quantum correlations for pure states. For mixed states the scenario is more subtle. In fact, separable mixed states may still contain features of quantumness, one of which is known as quantum discord [42, 43]. Quantum discord derives from the disagreement between the quantum generalization of two information quantities.

Classically, we know that the mutual information of two random variables A and B can be defined as

$$I(A : B) = H(A) + H(B) - H(A, B) , \quad (34)$$

where H is the Shannon entropy. Equivalently, using the chain rules for the entropy, we can write

$$I(A : B) = H(A) - H(A|B) , \quad (35)$$

where $H(\cdot|\cdot)$ is the conditional Shannon entropy.

In quantum mechanics, the previous quantities do not have equivalent generalizations. Given a bipartite quantum state ρ_{AB} , its quantum mutual information is expressed in terms of the von Neumann entropy as

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) . \quad (36)$$

This is the generalization of Eq. (34) and accounts for all the correlations present in the quantum state. Different is the generalization of Eq. (35) which is given by

$$C(\rho_{AB}) = S(\rho_A) - \inf_M H_M(A|B) , \quad (37)$$

where $M = \{M_k\}$ is a POVM acting on system B and

$$H_M(A|B) := \sum_k p_k S(\rho_{A|k}) , \quad (38)$$

where p_k is the probability of the outcome k and $\rho_{A|k}$ is the conditional state of A . The quantity $C(\rho_{AB})$ quantifies the classical correlations in the state corresponding to the amount of common randomness which can be extracted using one-way classical communication [44].

Quantum discord is defined as the difference between the total correlations and the classical correlations, i.e.

$$D(\rho_{AB}) = I(\rho_{AB}) - C(\rho_{AB}) . \quad (39)$$

Note that Eq. (37) is generally non-symmetric under system permutation. As a result, we have that the B -type quantities $C(\rho_{BA})$ and $D(\rho_{BA})$ are generally different from the A -type quantities of Eqs. (37) and (39).

For bosonic Gaussian states, we can define the so-called Gaussian quantum discord where the minimization is restricted to Gaussian POVMs. Consider a two-mode Gaussian state with CM in the blockform of Eq. (22). Then, classical correlations and Gaussian quantum discord are given by the formulas [45, 46]

$$C(\rho_{AB}) = h\left(\sqrt{\det \mathbf{A}}\right) - \Sigma , \quad (40)$$

and

$$D(\rho_{AB}) = h\left(\sqrt{\det \mathbf{B}}\right) - h(\nu_-) - h(\nu_+) + \Sigma , \quad (41)$$

where the expression of the term Σ is given in Ref. [46]. In our paper, we consider symmetric Gaussian states ($\mathbf{A} = \mathbf{B}$) so that there is no ambiguity in which type of discord and classical correlations we are using.

III. GAUSSIAN ENVIRONMENT WITH CORRELATED NOISE

In this section, we introduce and fully characterize our model of correlated Gaussian environment, which represents the simplest and most direct generalization of the standard memoryless Gaussian environment with losses and thermal noise.

Let us consider a Gaussian decoherence process which affects two bosonic modes, A and B , in a symmetric way. In order to describe the introduction of losses and thermal noise, we consider two beam splitters (with the same transmissivity τ) which combine modes A and B with two environmental modes, E_1 and E_2 , respectively. These ancillary modes are prepared in a zero-mean Gaussian state $\rho_{E_1 E_2}$ which is symmetric under E_1 - E_2 permutation. In the standard memoryless model depicted in Fig. 2(i), the environmental state is taken to be in a tensor-product $\rho_{E_1 E_2} = \rho \otimes \rho$, meaning that E_1 and E_2 are fully independent. In particular, ρ is a thermal state with CM $\omega \mathbf{I}$, where the noise variance $\omega = 2\bar{n} + 1$ quantifies the mean number of thermal photons \bar{n} entering the beam splitter. Each interaction is then equivalent to a lossy channel with transmissivity τ and thermal noise ω .

In our work, we generalize this Gaussian process to include the presence of correlations between the environmental modes as depicted in Fig. 2(ii). The simplest extension of the model consists of taking the ancillary modes, E_1 and E_2 , in a zero-mean Gaussian state $\rho_{E_1 E_2}$ with CM given by the symmetric normal form

$$\mathbf{V}_{E_1 E_2}(\omega, g, g') = \begin{pmatrix} \omega \mathbf{I} & \mathbf{G} \\ \mathbf{G} & \omega \mathbf{I} \end{pmatrix} , \quad (42)$$

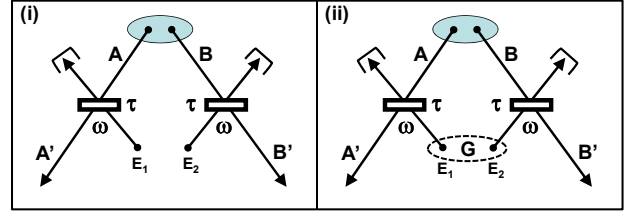


FIG. 2: Models of Gaussian environments. (i) Memoryless Gaussian environment with losses τ and thermal noise ω . (ii) Correlated Gaussian environment, with losses τ , thermal noise ω and correlations \mathbf{G} .

where $\omega \geq 1$ is the thermal noise variance associated with each ancilla, and the off-diagonal block

$$\mathbf{G} = \begin{pmatrix} g & \\ & g' \end{pmatrix} , \quad (43)$$

accounts for their correlations.

Clearly, when we consider the two interactions $A - E_1$ and $B - E_2$ separately, the environmental correlations are washed away. In fact, by tracing out E_2 , we are left with mode E_1 in a thermal state ($\mathbf{V}_{E_1} = \omega \mathbf{I}$) which is combined with mode A via the beam-splitter. In other words, we have again a lossy channel with transmissivity τ and thermal noise ω . The scenario is identical for the other mode B when we trace out E_1 . However, when we consider the joint action of the two environmental modes, the correlation block \mathbf{G} comes into play and the global dynamics of the two travelling modes becomes completely different from the standard memoryless scenario.

Before studying the system dynamics and the corresponding evolution of entanglement, we need to characterize the correlation block \mathbf{G} more precisely. In fact, the two correlation parameters, g and g' , cannot be completely arbitrary but must satisfy specific physical constraints. These parameters must vary within ranges which make the CM of Eq. (42) a bona-fide quantum CM. Given an arbitrary value of the thermal noise $\omega \geq 1$, the correlation parameters must satisfy the following three bona-fide conditions

$$|g| < \omega, \quad |g'| < \omega, \quad \omega^2 + gg' - 1 \geq \omega |g + g'| . \quad (44)$$

The proof of Eq. (44) is easy. We impose Eq. (24) to $\mathbf{V}_{E_1 E_2}(\omega, g, g')$. The positivity $\mathbf{V}_{E_1 E_2} > 0$ is equivalent to the positivity of the principal minors of the matrix. The positivity of the first two minors is trivially implied by $\omega \geq 1$. The third minor gives

$$\det \begin{pmatrix} \omega & 0 & g \\ 0 & \omega & 0 \\ g & 0 & \omega \end{pmatrix} > 0 \Leftrightarrow \omega(\omega^2 - g^2) > 0, \quad (45)$$

which is equivalent to

$$|g| < \omega . \quad (46)$$

The fourth minor corresponds to the determinant

$$\det \mathbf{V}_{E_1 E_2} = (\omega^2 - g^2)(\omega^2 - g'^2), \quad (47)$$

and its positivity $\det \mathbf{V}_{E_1 E_2} > 0$ leads to the condition

$$|g'| < \omega. \quad (48)$$

Finally, using $\Delta = 2(\omega^2 + gg')$ and Eq. (47), we find that $\nu_-^2 \geq 1$ is equivalent to [47]

$$\omega^2 + gg' - 1 \geq \omega |g + g'|. \quad (49)$$

In conclusion, an environmental CM of the form (42) with thermal noise $\omega \geq 1$ is a bona-fide CM if the correlation parameters g and g' satisfy the three bona-fide conditions of Eq. (44). Using Eqs. (19) and (47), we also see that its purity is given by

$$\mu_{E_1 E_2} = [(\omega^2 - g^2)(\omega^2 - g'^2)]^{-1/2}. \quad (50)$$

A. Separability properties

Once that we have fully clarified the bona-fide conditions for the environment, the next step is to characterize its separability properties.

For this aim, we compute the smallest PTS eigenvalue associated with $\mathbf{V}_{E_1 E_2}$. After simple algebra, we get

$$\varepsilon = \sqrt{\omega^2 - gg' - \omega |g - g'|}. \quad (51)$$

Provided that Eq. (44) is satisfied, the separability condition $\varepsilon \geq 1$ is equivalent to

$$\omega^2 - gg' - 1 \geq \omega |g - g'|. \quad (52)$$

The various conditions of bona-fide and separability can be combined together. An environment of the form (42) with thermal noise $\omega \geq 1$ is bona-fide and separable when the correlation parameters satisfy

$$|g| < \omega, \quad |g'| < \omega, \quad \omega^2 - 1 \geq \max\{\Gamma_-, \Gamma_+\}. \quad (53)$$

where

$$\Gamma_- := \omega |g + g'| - gg', \quad (54)$$

$$\Gamma_+ := \omega |g - g'| + gg'. \quad (55)$$

By contrast, it is bona-fide and entangled when

$$|g| < \omega, \quad |g'| < \omega, \quad \Gamma_- \leq \omega^2 - 1 < \Gamma_+. \quad (56)$$

To better clarify the structure of the environment, we provide a numerical example in Fig. 3. In this figure, we consider the *correlation plan* which is spanned by the two parameters g and g' . For a given value of the thermal noise ω , we identify the subset of points which satisfy the bona-fide conditions of Eq. (44). This subset corresponds to the white area in the figure. Within this area, we then characterize the regions which correspond to separable environment (area labelled by S) and entangled environment (areas labelled by E).

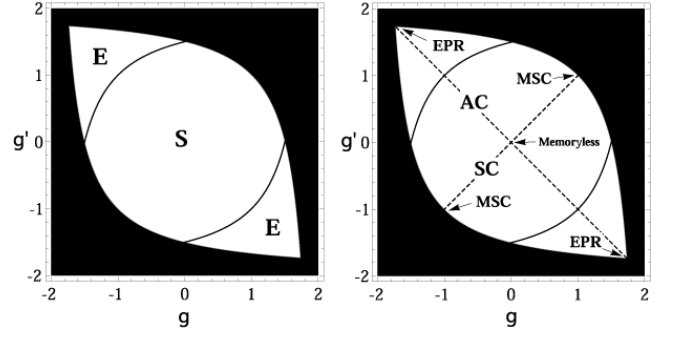


FIG. 3: *Left.* Correlation plan (g, g') for the Gaussian environment, corresponding to thermal noise $\omega = 2$. The black area identifies forbidden environments (correlations are too strong to be compatible with quantum mechanics). White area identifies physical environments, i.e., the subset of points which satisfy the bona-fide conditions of Eq. (44). Within this area, the inner region labelled by S identifies separable environments [Eq. (53) is satisfied] while the two outer regions identify entangled environments [Eq. (56) is satisfied]. *Right.* Correlation plan displaying the SC environments (bisector $g' = g$) and the AC environments (bisector $g' = -g$). The MSC and EPR environments are also displayed (extremal points on the bisectors).

B. Specific types of environments

Here we specify our model of Gaussian environment to particular cases with specific correlation properties. These types of environment will be taken as examples for deriving analytical results.

Before discussing these cases, it is trivial to say that, for $g' = g = 0$, our environment collapses into a memoryless Gaussian environment, described by two identical lossy channels with transmissivity τ and thermal noise ω . This is represented by the origin in Fig. 3. Also note that the condition $\omega = 1$ corresponding to vacuum noise is only compatible with this environment. In other words, for $\omega = 1$, we must necessarily have $g' = g = 0$, since this is the unique solution which is compatible with the bona-fide conditions of Eq. (44).

The first type of correlated environment, that we call “symmetrically correlated” (SC) corresponds to taking $g' = g$. This means that positions and momenta of the two environmental modes are correlated in exactly the same way. On the correlation plan, this environment is represented by one of the points lying on the bisector of the first and third quadrants (see Fig. 3).

It is easy to check that the bona-fide conditions of Eq. (44) simplify to the unique inequality

$$|g| \leq \omega - 1, \quad (57)$$

so that the maximal symmetrical correlations (MSC) are given by $g = \omega - 1$ or $g = 1 - \omega$, which are the two extremal points indicated in Fig. 3. As we can see from the figure, the SC environments are always separable, so that E_1 and E_2 are correlated but not entangled. In

fact, the separability condition of Eq. (52) becomes $|g| \leq \sqrt{\omega^2 - 1}$ which is always satisfied by $\omega \geq 1$ and Eq. (57).

The SC environment is separable but generally mixed, since Eq. (50) becomes $\mu_{E_1 E_2} \leq (2\omega - 1)^{-1}$ which is less than 1 for any $\omega > 1$. It is pure only for $\omega = 1$ when it collapses into a memoryless environment. Using Eqs. (40) and (41), we investigate the nature of its separable correlations in terms of classical correlations and quantum discord. The amount of classical correlations is equal to

$$C = h(\omega) - h\left(\omega - \frac{g^2}{\omega + 1}\right), \quad (58)$$

while the Gaussian quantum discord is given by

$$D = h(\omega) - h(\omega - g) - h(\omega + g) + h\left(\omega - \frac{g^2}{\omega + 1}\right). \quad (59)$$

The second type of environment, that we call “asymmetrically correlated” (AC) corresponds to the condition $g' = -g$. This means that positions and momenta have opposite correlations, i.e., if positions are correlated (anticorrelated), then momenta are anticorrelated (correlated). On the correlation plan, this environment is represented by one of the points lying on the bisector of the second and fourth quadrants (see Fig. 3).

In this case, the bona-fide conditions of Eq. (44) simplify to the inequality

$$|g| \leq \sqrt{\omega^2 - 1}. \quad (60)$$

For maximal correlations $|g| = \sqrt{\omega^2 - 1}$ we have an EPR state, which is pure and maximally entangled. Depending on the sign of g , we have two different EPR environments: The positive EPR environment ($g = \sqrt{\omega^2 - 1}$) with positions correlated, and the negative EPR environment ($g = -\sqrt{\omega^2 - 1}$) with positions anticorrelated.

In general, the AC environment can be separable or entangled, as evident from Fig. 3. By using Eq. (52), we see that it is separable for

$$|g| \leq \omega - 1, \quad (61)$$

while it is entangled for stronger correlations

$$\omega - 1 < |g| \leq \sqrt{\omega^2 - 1}. \quad (62)$$

Finally, we can compute the amount of classical correlations and Gaussian quantum discord. These quantities are respectively given by Eq. (58) and

$$D = h(\omega) - 2h\left(\sqrt{\omega^2 - g^2}\right) + h\left(\omega - \frac{g^2}{\omega + 1}\right). \quad (63)$$

IV. DIRECT DISTRIBUTION OF ENTANGLEMENT

Let us study the system dynamics and the entanglement propagation in the presence of a correlated Gaussian environment. Suppose that Charlie has an entanglement source described by an EPR state ρ_{AB} with CM

$\mathbf{V}(\mu)$ given in Eq. (29). We may consider the different scenarios depicted in Fig. 4. Charlie may attempt to distribute entanglement to Alice and Bob as shown in Fig. 4(i), or he may try to share entanglement with one of the remote parties, as shown in Figs. 4(ii) and (iii).

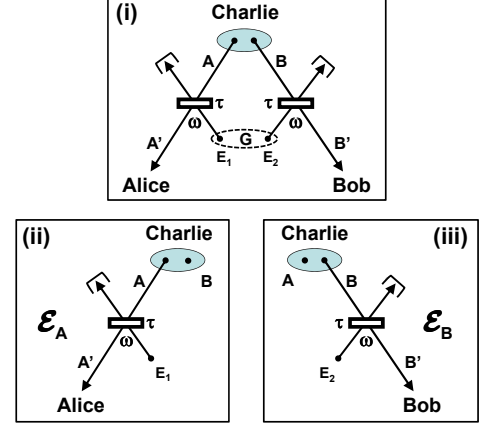


FIG. 4: Scenarios for direct distribution of entanglement. (i) Charlie has two modes A and B prepared in an EPR state ρ_{AB} . In order to distribute entanglement to the remote parties, Charlie transmits the two modes through the correlated Gaussian environment characterized by transmissivity τ , thermal noise ω and correlations \mathbf{G} . (ii) Charlie aims to share entanglement with Alice. He then keeps mode B while sending mode A to Alice through the lossy channel \mathcal{E}_A . (iii) Charlie aims to share entanglement with Bob. He then keeps mode A while sending mode B to Bob through the lossy channel \mathcal{E}_B .

Let us start considering the scenario where Charlie aims to share entanglement with one of the remote parties (one-mode transmission). In particular, suppose that Charlie wants to share entanglement with Bob (by symmetry the derivation is the same if we consider Alice). For sharing entanglement, Charlie keeps mode A while sending mode B to Bob as shown in Fig. 4(iii). The action of the environment is therefore reduced to $\mathcal{I}_A \otimes \mathcal{E}_B$, where \mathcal{E}_B is a lossy channel applied to mode B . It is easy to check that the output state $\rho_{AB'}$ shared by Charlie and Bob is Gaussian with zero mean and CM

$$\mathbf{V}_{AB'} = \begin{pmatrix} \mu \mathbf{I} & \mu' \sqrt{\tau} \mathbf{Z} \\ \mu' \sqrt{\tau} \mathbf{Z} & x \mathbf{I} \end{pmatrix}, \quad (64)$$

where

$$x := \tau\mu + (1 - \tau)\omega. \quad (65)$$

In fact, the global input state $\rho_{AB} \otimes \rho_{E_2}$ has CM $\mathbf{V}(\mu) \oplus \omega \mathbf{I}$. This CM is subject to the symplectic transformation $\mathbf{I} \oplus \mathbf{S}(\tau)$, where $\mathbf{S}(\tau)$ is the beam splitter matrix of Eq. (11) acting on modes B and E_2 . Tracing out the environment from the output CM, we get Eq. (64).

Remarkably, we can compute closed analytical formulas in the limit of large μ , i.e., large input entanglement.

In this case, the entanglement of the output state is quantified by the PTS eigenvalue

$$\varepsilon = \frac{1 - \tau}{1 + \tau} \omega . \quad (66)$$

The EB condition corresponds to $\varepsilon \leq 1$, which provides

$$\omega \geq \frac{1 + \tau}{1 - \tau} := \omega_{\text{EB}} , \quad (67)$$

or equivalently

$$\bar{n} \geq \frac{\tau}{1 - \tau} . \quad (68)$$

Despite the EB condition of Eq. (67) has been derived for an EPR input, it is actually completely general. In other words, a lossy channel \mathcal{E}_B with transmissivity τ and thermal noise $\omega \geq \omega_{\text{EB}}$ destroys the entanglement whatever Charlie's input state ρ_{AB} is. In fact, Eq. (67) corresponds exactly to the general EB condition for lossy channels which has been derived in Ref. [14]. It is therefore clear that the threshold condition $\omega = \omega_{\text{EB}}$ guarantees one-mode EB, i.e., the impossibility for Charlie to share entanglement with Alice or Bob.

Now the central question is the following: Suppose that Charlie cannot share entanglement with the remote parties (one-mode EB), can Charlie still distribute entanglement to them? In other words, suppose that the correlated Gaussian environment has transmissivity τ and thermal noise $\omega = \omega_{\text{EB}}$, so that the lossy channels \mathcal{E}_A and \mathcal{E}_B are EB. Is it still possible to use the joint channel \mathcal{E}_{AB} to distribute entanglement to Alice and Bob? In the following, we explicitly answer to this question. Furthermore, we will show that the distributed entanglement can also be distilled by one-way protocols.

Let us derive the general evolution of the two modes A and B under the action of the environment [see Fig. 4(i)]. Since the input EPR state ρ_{AB} is Gaussian and the environment is Gaussian, the output state $\rho_{A'B'}$ is also Gaussian. This state has zero mean and CM given by

$$\mathbf{V}_{A'B'} = \tau \mathbf{V}_{AB} + (1 - \tau) \mathbf{V}_{E_1 E_2} = \begin{pmatrix} x\mathbf{I} & \mathbf{H} \\ \mathbf{H} & x\mathbf{I} \end{pmatrix} , \quad (69)$$

where

$$\mathbf{H} := \tau \mu' \mathbf{Z} + (1 - \tau) \mathbf{G} . \quad (70)$$

For large μ , we can easily derive the symplectic spectrum

$$\nu_{\pm} = \sqrt{(2\omega + g' - g \pm |g + g'|)(1 - \tau)\tau\mu} , \quad (71)$$

and the smallest PTS eigenvalue

$$\varepsilon = (1 - \tau) \sqrt{(\omega - g)(\omega + g')} , \quad (72)$$

which quantifies the amount of entanglement distributed to Alice and Bob. In the same limit, we can easily compute the coherent information between the two remote

parties. In fact, using the formula of Eq. (28) for diverging spectra, we get

$$I(A)B = \log \frac{1}{e\varepsilon} . \quad (73)$$

Thus, remote entanglement is distributed for $\varepsilon < 1$ and this entanglement is distillable for $\varepsilon < e^{-1}$.

Suppose that the environment has thermal noise $\omega = \omega_{\text{EB}}$ (one-mode EB). Then, we can write

$$\begin{aligned} \varepsilon &= \sqrt{[1 + \tau - (1 - \tau)g][1 + \tau + (1 - \tau)g']} \\ &:= \varepsilon(\tau, g, g') \end{aligned} \quad (74)$$

Answering our previous question corresponds to checking the existence of environmental parameters τ , g and g' , for which $\varepsilon < 1$. For a given value of the transmissivity τ , we look for regions in the correlation plan (g, g') where remote entanglement is distributed ($\varepsilon < 1$) and possibly distillable ($\varepsilon < e^{-1}$). This is done in Fig. 5 for several numerical values of the transmissivity.

In Fig. 5, the environments identified by the gray region, that we call the “activation area”, allow Charlie to distribute entanglement to Alice and Bob ($\varepsilon < 1$), despite it is impossible for him to share entanglement with the remote parties. In other words, these environments are two-mode entanglement preserving (EP), despite they are one-mode EB. Furthermore, we can also identify environments which are able to generate distillable entanglement ($\varepsilon < e^{-1}$).

The most remarkable feature in Fig. 5 is represented by the presence of separable environments in the activation area. In other words, there are separable environments which contain enough correlations to activate the distribution of entanglement to Alice and Bob. Furthermore, for sufficiently high transmissivities and correlations, this entanglement can also be distilled by means of one-way protocols. Also note from Fig. 5 that the weight of separable environments in the activation area increases for increasing transmissivities, with the entangled environments almost disappearing for $\tau = 0.9$.

It is important to stress that achieving the simultaneous conditions of one-mode EB and two-mode EP is not surprising if we consider entangled environments. For instance, we may consider two beam-splitters with zero transmissivity, so that Charlie's state is completely reflected into the environment and the (entangled) state of the environment is reflected to Alice and Bob. This scenario is certainly one-mode EB since Charlie has no chance of sharing part of his state with Alice or Bob, and it is also two-mode EP since the loss of Charlie's initial entanglement is replaced by the injection of entanglement from the environment, which is then distributed to Alice and Bob.

By contrast, achieving the conditions of one-mode EB and two-mode EP with separable environments is surprising, because no injection of entanglement is present but still the environmental correlations are strong enough to restore the broken entanglement.

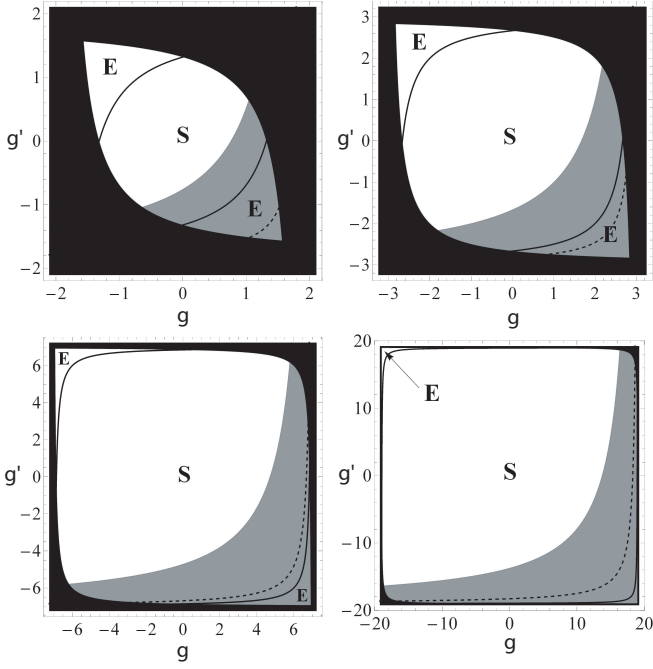


FIG. 5: Analysis of the remote entanglement on the correlation plan for different values of the transmissivity $\tau = 0.3, 0.5, 0.75$, and 0.9 (from top left to bottom right). Corresponding values of the thermal noise are determined by the one-mode EB condition $\omega = \omega_{EB}$. In each inset, the non-black area identifies the set of physical environments, which are divided into separable (S) and entangled (E) environments. The gray region is the activation area and identifies those environments which enable Charlie to distribute entanglement to Alice and Bob ($\varepsilon < 1$). Within the activation area, the environments below the dashed curve enable Charlie to distribute distillable entanglement to the remote parties ($\varepsilon < e^{-1}$).

A. Direct distribution in specifically correlated environments

Here we analyze the scheme of direct distribution in the presence of the specific environments discussed in Sec. III B, i.e., the SC and AC environments, corresponding to the bisectors of the correlation plan. We consider these environments at the EB threshold $\omega = \omega_{EB}$, so that they are one-mode EB. In the limit of large μ , we derive the regimes of parameters for which remote entanglement can be distributed and distilled.

Let us start with the SC environment ($g' = g$) which is always separable. In this case, the remote entanglement of Alice and Bob is quantified by $\varepsilon = \varepsilon(\tau, g, g)$ according to Eq. (74). This quantity takes the optimal value

$$\varepsilon_{MSC} = \sqrt{(1 - \tau)(1 + 3\tau)} \quad (75)$$

when the environment is MSC, i.e., it has maximal correlation parameter

$$|g_{MSC}| = \omega_{EB} - 1 = \frac{2\tau}{1 - \tau}. \quad (76)$$

From Eq. (75) we can easily check that remote entanglement is generated ($\varepsilon < 1$) only if $\tau > 2/3$, and remote entanglement can be distilled ($\varepsilon < e^{-1}$) only if $\tau \gtrsim 0.96$.

For an arbitrary SC environment, remote entanglement is generated for $\tau > 2/3$ and

$$|g_{EG}| < |g| \leq |g_{MSC}|, \quad (77)$$

where the entanglement generation bound is given by

$$|g_{EG}| := \frac{\sqrt{\tau(\tau + 2)}}{1 - \tau}. \quad (78)$$

Entanglement is then distillable when $\tau \gtrsim 0.96$ and

$$|g_{ED}| < |g| \leq |g_{MSC}|. \quad (79)$$

where the entanglement distillation bound is equal to

$$|g_{ED}| := \frac{\sqrt{e^2(1 + \tau)^2 - 1}}{e(1 - \tau)} \geq |g_{EG}|. \quad (80)$$

See Fig. 6 for a pictorial representation.

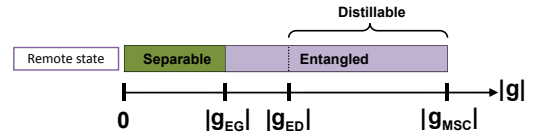


FIG. 6: Schematic representation showing the separability properties of the remote state in terms of the correlation parameter of the SC environment. The environment is separable and satisfies the one-mode EB condition $\omega = \omega_{EB}$. Here we consider high transmissivity $\tau \gtrsim 0.96$ so that remote entanglement can also be distilled.

Now consider the AC environment ($g' = -g$) which is separable for $|g| \leq |g_S|$ and entangled for $|g_S| \leq |g| \leq |g_{EPR}|$, where

$$|g_S| := \omega_{EB} - 1, |g_{EPR}| := \sqrt{\omega_{EB}^2 - 1} = \frac{2\sqrt{\tau}}{1 - \tau}. \quad (81)$$

Alice and Bob's remote entanglement is quantified by

$$\varepsilon = \varepsilon(\tau, g, -g) = 1 + \tau - (1 - \tau)g, \quad (82)$$

which can be less than one only for positive values of the correlation parameter g . This asymmetry, which is also evident from Fig. 5, depends on the fact that Charlie's input state has EPR correlations of the type $\hat{q}_A = \hat{q}_B$ and $\hat{p}_A = -\hat{p}_B$. These correlations tend to be preserved by AC environments with positive g (having correlations of the same type) while they tend to be destroyed by AC environments with negative g (having correlations of the opposite type).

The optimal distribution of entanglement is clearly achieved for the positive EPR environment $g = |g_{EPR}|$, for which we have $\varepsilon_{EPR} = (1 - \sqrt{\tau})^2$. The optimal distribution in a separable AC environment is achieved at the border value $g = |g_S|$ where $\varepsilon_S = 1 - \tau$. From the

expression of Eq. (82), we see that remote entanglement is generated at any transmissivity τ for values of the correlation parameter

$$g_{EG} < g \leq |g_{EPR}|, \quad (83)$$

where the entanglement generation bound is equal to

$$g_{EG} := \frac{\tau}{1 - \tau}. \quad (84)$$

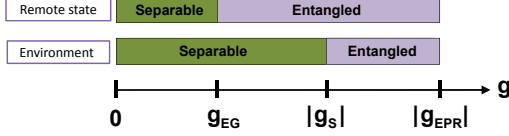


FIG. 7: Schematic representation showing the separability properties of the remote state versus those of the AC environment in terms of the correlation parameter. The environment satisfies the one-mode EB condition $\omega = \omega_{EB}$.

As shown in Fig. 7 the generation of remote entanglement is possible in the presence of separable environments ($g_{EG} \leq |g_S|$). Entanglement distillation is possible for $\tau > e^{-1}(\sqrt{e} - 1)^2 \simeq 0.15$ and correlation values

$$g_{ED} < g \leq |g_{EPR}|, \quad (85)$$

where the entanglement distillation bound is given by

$$g_{ED} := \frac{1 + \tau - e^{-1}}{1 - \tau} \geq g_{EG}. \quad (86)$$

At higher transmissivities $\tau > 1 - e^{-1} \simeq 0.63$ we have that distillation is possible even for separable environments ($g_{ED} \leq |g_S|$). For a schematic see Fig. 8.

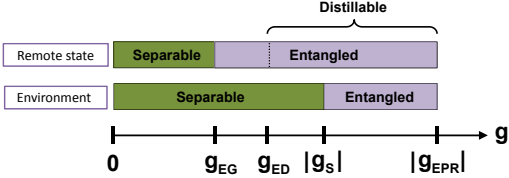


FIG. 8: Schematic representation showing the separability properties of the remote state versus those of the AC environment in terms of the correlation parameter. The environment satisfies the one-mode EB condition $\omega = \omega_{EB}$. Here we consider high transmissivity $\tau \gtrsim 0.63$ so that remote entanglement is distillable even in the presence of separable environments.

Finally, it is interesting to investigate what kind of correlations are present in separable environments which are one-mode EB and two-mode EP. This is done in Fig. 9 for the SC and AC environments. Remote entanglement is generated ($\varepsilon < 1$) when purely-classical correlations C are appreciably different from zero. The role of Gaussian quantum discord D seems to become more important for high values of g , in particular for the AC environment, where D approaches C when the environment becomes entangled, reaching the extreme value $D = C$ for the EPR environments.

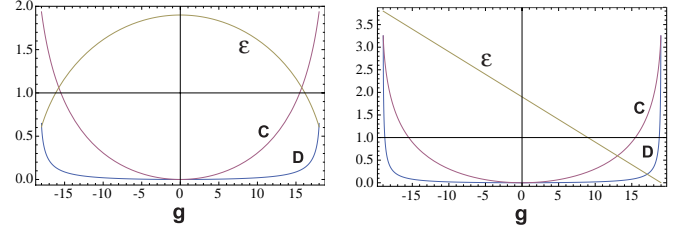


FIG. 9: Remote entanglement ε , Gaussian quantum discord D and purely-classical correlations C are plotted as functions of the correlation parameter g , for the SC environment (left) and the AC environment (right). Transmission is $\tau = 0.9$ and thermal noise is $\omega_{EB} = 19$ (one-mode EB). The SC environment is always separable while the AC environment is separable for $|g| \leq 18$ and entangled for $18 < |g| \leq 18.97$. Remote entanglement is generated ($\varepsilon < 1$) when C is appreciably different from zero. Remote entanglement is optimal for the MSC environments in the left panel, and the positive EPR environment in the right panel.

B. Evolution of the EPR correlations

In order to give another point of view to the dynamics of the process, we describe here the evolution of the EPR correlations under the effect of the correlated Gaussian environment [see Fig. 4(i)]. We can easily write the input-output Bogoliubov transformations

$$\hat{\mathbf{x}}_{A'} = \sqrt{\tau} \hat{\mathbf{x}}_A + \sqrt{1 - \tau} \hat{\mathbf{x}}_{E_1} \quad (87)$$

$$\hat{\mathbf{x}}_{B'} = \sqrt{\tau} \hat{\mathbf{x}}_B + \sqrt{1 - \tau} \hat{\mathbf{x}}_{E_2} \quad (88)$$

where $\hat{\mathbf{x}} = (q, p)^T$ is a vector of quadratures. From these equations, we can extract the output EPR operators

$$\hat{q}'_- := \frac{\hat{q}_{A'} - \hat{q}_{B'}}{\sqrt{2}}, \quad \hat{p}'_+ = \frac{\hat{p}_{A'} + \hat{p}_{B'}}{\sqrt{2}}. \quad (89)$$

Now, using the CM of the input EPR state and that of the environment, we can compute the variances

$$\begin{aligned} \mathbf{\Lambda} &:= \begin{pmatrix} V(\hat{q}'_-) & \\ & V(\hat{p}'_+) \end{pmatrix} \\ &= \tau(\mu - \mu')\mathbf{I} + (1 - \tau)(\omega\mathbf{I} - \mathbf{Z}\mathbf{G}). \end{aligned} \quad (90)$$

In the limit of $\mu \gg 1$, we have

$$\mathbf{\Lambda} \rightarrow \mathbf{\Lambda}_\infty = (1 - \tau)(\omega\mathbf{I} - \mathbf{Z}\mathbf{G}), \quad (91)$$

and assuming the EB condition $\omega = \omega_{EB}$ we get

$$\mathbf{\Lambda}_{\infty, EB} = (1 + \tau)\mathbf{I} - (1 - \tau)\mathbf{Z}\mathbf{G}. \quad (92)$$

Analyzing Eq. (92) we may guess the possibility of realizing the EPR condition $\mathbf{\Lambda}_{\infty, EB} < \mathbf{I}$ by adopting suitable choices of the correlation block \mathbf{G} .

Let us explicitly compare the different types of environments. For the memoryless environment ($\mathbf{G} = \mathbf{0}$) we have $\mathbf{\Lambda}_{\infty, EB} = (1 + \tau)\mathbf{I}$ which means that the EPR variances are always greater than or equal to one, i.e., EPR correlations do not survive.

For the AC environment ($\mathbf{G} = g\mathbf{Z}$) we have $\Lambda_{\infty,EB} = [(1 + \tau) - (1 - \tau)g]\mathbf{I}$. It is easy to check that the EPR condition $\Lambda_{\infty,EB} < \mathbf{I}$ is achieved by physical values of $g > g_{EG}$, where g_{EG} is given in Eq. (84). This means that the remote entanglement generated by this environment is always in the form of EPR correlations (of the same type of the original EPR state at Charlie's station).

For the SC environment ($\mathbf{G} = g\mathbf{I}$) we have $\Lambda_{\infty,EB} = (1 + \tau)\mathbf{I} - (1 - \tau)g\mathbf{Z}$ and we can check that the condition $\Lambda_{\infty,EB} < \mathbf{I}$ is not realizable by any choice of g . In particular, for the MSC environment $g = \omega_{EB} - 1$ we have

$$\Lambda_{\infty,EB} = \begin{pmatrix} 1 - \tau & \\ & 1 + 3\tau \end{pmatrix}. \quad (93)$$

Thus, we see that the initial EPR correlations do not survive in this case. Nevertheless remote entanglement can be distributed in the presence of this environment.

V. INDIRECT DISTRIBUTION OF ENTANGLEMENT

In this section we consider the indirect distribution of entanglement, i.e., the protocol of entanglement swapping. We start with a brief review of this protocol in the ideal case of no noise. Then, we generalize its theory to the case of correlated-noise Gaussian environments, where we prove how entanglement swapping can be activated in the presence of one-mode EB.

A. Entanglement swapping in the absence of noise

Consider two remote parties, Alice and Bob, who possess two identical EPR states with CM given in Eq. (29). At Alice's station, the EPR state describes modes a and A , while at Bob's station it describes modes b and B . Alice and Bob keep modes a and b , while sending modes A and B to Charlie, where a Bell measurement is performed. This means that the travelling modes A and B are combined in a balanced beam splitter whose output modes “−” and “+” are homodyned, with mode “−” measured in the position quadrature and mode “+” in the momentum quadrature. In other words, Charlie measures the two EPR quadratures \hat{q}_- and \hat{p}_+ which are defined by Eq. (32). The Bell measurement provides two outcomes, q_- and p_+ , which can be compacted into a single complex variable $\gamma := q_- + ip_+$. The classical variable γ is finally communicated to Alice and Bob, with the result of projecting their remote modes a and b into a conditional state $\rho_{ab|\gamma}$ (see Fig. 10).

Since the input states are pure Gaussian and the Bell measurement is a Gaussian measurement which projects pure states into pure states, we have that the remote conditional state $\rho_{ab|\gamma}$ turns out to be a pure Gaussian state. This state has a measurement-dependent mean $\mathbf{x} = \mathbf{x}(\gamma)$

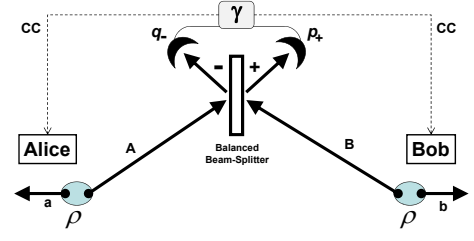


FIG. 10: Entanglement swapping in the absence of noise. See text for explanation.

which can always be deleted by Alice and Bob via conditional displacements. It is clear that these local unitaries do not alter the amount of entanglement in the state, as long as they are perfectly implemented (see Ref. [48] for a general analysis which includes imperfect displacements). The conditional CM $\mathbf{V}_{ab|\gamma}$ can be computed using the input-output formula for Gaussian entanglement swapping which has been proven in Ref. [49]. We get

$$\mathbf{V}_{ab|\gamma} = \frac{1}{2\mu} \begin{pmatrix} (\mu^2 + 1)\mathbf{I} & (\mu^2 - 1)\mathbf{Z} \\ (\mu^2 - 1)\mathbf{Z} & (\mu^2 + 1)\mathbf{I} \end{pmatrix}. \quad (94)$$

Its smallest PTS eigenvalue is equal to $\varepsilon = \mu^{-1}$, which means that remote entanglement is always generated for entangled inputs ($\mu > 1$). Furthermore, remote entanglement is present in the form of EPR correlations since the two EPR quadratures $\hat{q}_- = (\hat{q}_a - \hat{q}_b)/\sqrt{2}$ and $\hat{p}_+ = (\hat{p}_a + \hat{p}_b)/\sqrt{2}$ have variances

$$V(\hat{q}_-) = V(\hat{p}_+) = \mu^{-1}. \quad (95)$$

The simplest description of the entanglement swapping protocol can be given when we consider the limit for $\mu \rightarrow \infty$. In this case the initial states are ideal EPR states with quadratures perfectly correlated, i.e., $\hat{q}_a = \hat{q}_A$ and $\hat{p}_a = -\hat{p}_A$ for Alice, and $\hat{q}_b = \hat{q}_B$ and $\hat{p}_b = -\hat{p}_B$ for Bob. Then, the overall action of Charlie, i.e., the Bell measurement plus classical communication, corresponds to create a remote state with

$$\hat{q}_b = \hat{q}_a - \sqrt{2}q_-, \quad \hat{p}_b = -\hat{p}_a - \sqrt{2}p_+. \quad (96)$$

The quadratures of the two remote modes are perfectly correlated, up to an erasable displacement. In other words, the ideal EPR correlations have been swapped from the initial states to the final conditional state $\rho_{ab|\gamma}$.

B. Entanglement swapping in a correlated Gaussian environment

The theory of entanglement swapping can be extended to include the presence of loss and correlated noise. We consider our model of correlated Gaussian environment with transmission τ , thermal noise ω and correlations \mathbf{G} . The modified scenario is depicted in Fig. 11.

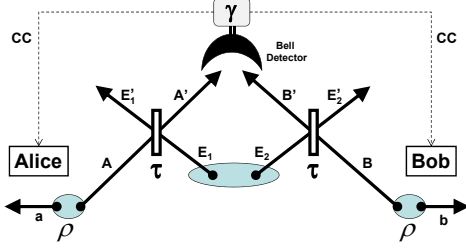


FIG. 11: Entanglement swapping in the presence of loss, thermal noise and environmental correlations (correlated Gaussian environment). The Bell detector has been simplified.

1. Swapping of EPR correlations

For simplicity, we start by studying the evolution of the EPR correlations under ideal input conditions ($\mu \rightarrow +\infty$). The quadratures of the travelling modes, A and B , are transformed according to the same input-output relations specified in Eqs. (87) and (88). Then, the output modes A' and B' are measured by Charlie, while modes E'_1 and E'_2 are traced out. After the classical communication of the outcome γ , the quadratures of the remote modes a and b satisfy the asymptotic relations

$$\hat{q}_b = \hat{q}_a - \sqrt{\frac{2}{\tau}} \left(q_- - \sqrt{1-\tau} \hat{\delta}_q \right), \quad (97)$$

$$\hat{p}_b = -\hat{p}_a - \sqrt{\frac{2}{\tau}} \left(p_+ - \sqrt{1-\tau} \hat{\delta}_p \right), \quad (98)$$

where $\hat{\delta}_q = (\hat{q}_{E_1} - \hat{q}_{E_2})/\sqrt{2}$ and $\hat{\delta}_p = (\hat{p}_{E_1} + \hat{p}_{E_2})/\sqrt{2}$ are noise variables introduced by the environment.

Considering the EPR quadratures \hat{q}_- and \hat{p}_+ , and averaging them over the states, we compute the asymptotic EPR variances

$$\Lambda := \begin{pmatrix} V(\hat{q}_-) \\ V(\hat{p}_+) \end{pmatrix} \rightarrow \Lambda_\infty = \frac{1-\tau}{\tau} (\omega \mathbf{I} - \mathbf{ZG}). \quad (99)$$

Assuming the EB condition $\omega = \omega_{EB}$, we finally get

$$\Lambda_{\infty,EB} = \frac{1}{\tau} [(1+\tau)\mathbf{I} - (1-\tau)\mathbf{ZG}], \quad (100)$$

which is equal to Eq. (92) up to a factor τ^{-1} .

In the case of a memoryless environment ($\mathbf{G} = \mathbf{0}$) we get $\Lambda_{\infty,EB} = (1+\tau^{-1})\mathbf{I} \geq \mathbf{I}$, which means that the EPR correlations cannot be swapped to the remote systems. However, it is evident from Eq. (100) that there are choices for the correlation block \mathbf{G} such that the EPR condition $\Lambda_{\infty,EB} < \mathbf{I}$ is satisfied. For instance, this happens when we consider the AC environment ($\mathbf{G} = g\mathbf{Z}$). In this case it is easy to check that $\Lambda_{\infty,EB} < \mathbf{I}$ is satisfied for $\tau \geq 1/4$ and

$$(1-\tau)^{-1} < g \leq |g_{EPR}|. \quad (101)$$

Under these conditions, the original EPR correlations are successfully swapped to the remote modes. In particular,

for $\tau > 1/2$ and $(1-\tau)^{-1} < g \leq |g_S|$ there are separable AC environments which do the job. Thus, despite the presence of one-mode EB, the injection of separable correlations from the environment reactivates the swapping protocol generating remote EPR correlations.

Finally, we have that SC environments ($\mathbf{G} = g\mathbf{I}$) never satisfy the EPR condition, so that the initial EPR correlations are always lost. However, as we shall see below, quantum entanglement can be swapped with success.

2. Swapping and distillation of entanglement

Here we discuss in detail how quantum entanglement can be distributed by the swapping protocol in the presence of a correlated Gaussian environment. In particular, we aim to address the following questions:

- Suppose that Alice and Bob are not able to share entanglement with Charlie because the environment is one-mode EB, is it still possible for Charlie to distribute entanglement to the remote parties by exploiting the environmental correlations injected in the swapping protocol?
- In particular, is the swapping successful when the environmental correlations are separable?
- Finally, can Alice and Bob distill the swapped entanglement by one-way protocols?

Our previous discussion on EPR correlations suggests that these questions have positive answers. Here we explicitly show this is specifically true for quantum entanglement by finding the corresponding good regimes of parameters for the Gaussian environment.

In order to study the propagation of entanglement we first need to derive the CM $\mathbf{V}_{ab|\gamma}$ of the conditional remote state $\rho_{ab|\gamma}$. As before, we have two identical EPR states at Alice's and Bob's stations with CM $\mathbf{V}(\mu)$ given in Eq. (29). The travelling modes A and B are sent to Charlie through a Gaussian environment with transmissivity τ , thermal noise ω and correlations \mathbf{G} . After the Bell measurement and the classical communication of the result γ , the conditional remote state at Alice's and Bob's stations is Gaussian with CM

$$\mathbf{V}_{ab|\gamma} = \begin{pmatrix} \mu \mathbf{I} & \\ & \mu \mathbf{I} \end{pmatrix} - \frac{(\mu^2 - 1)\tau}{2} \begin{pmatrix} \frac{1}{\theta} & & -\frac{1}{\theta} \\ & \frac{1}{\theta'} & \\ -\frac{1}{\theta} & & \frac{1}{\theta} \\ & \frac{1}{\theta'} & \\ & & \frac{1}{\theta'} \end{pmatrix}, \quad (102)$$

where

$$\theta = \tau\mu + (1-\tau)(\omega - g), \quad \theta' = \tau\mu + (1-\tau)(\omega + g'). \quad (103)$$

(See Appendix A for its derivation).

From the CM of Eq. (102) we derive the smallest PTS eigenvalue ε quantifying the remote entanglement at Alice's and Bob's stations. In the limit of large input entanglement $\mu \gg 1$, we find a closed formula in terms of

the environmental parameters, i.e.,

$$\varepsilon = \frac{1-\tau}{\tau} \sqrt{(\omega-g)(\omega+g')} := \varepsilon(\tau, \omega, g, g'), \quad (104)$$

which is equal to Eq. (72) up to a factor τ^{-1} . As before, this eigenvalue not only determines the log-negativity but also the coherent information associated with the remote state $\rho_{ab|\gamma}$. In fact, in the limit of large μ , the determinant of the CM (102) becomes

$$\det \mathbf{V}_{ab|\gamma} = \left[\frac{(1-\tau)\mu}{\tau} \right]^2 (\omega-g)(\omega+g') = (\varepsilon\mu)^2. \quad (105)$$

Then, the reduced CM of Bob

$$\mathbf{V}_{b|\gamma} = \mu \mathbf{I} - \frac{(\mu^2 - 1)\tau}{2} \begin{pmatrix} \frac{1}{\theta} & \\ & \frac{1}{\theta'} \end{pmatrix}, \quad (106)$$

has asymptotic determinant equal to $\det \mathbf{V}_{b|\gamma} = \mu^2$. One can easily check that the symplectic spectra of $\mathbf{V}_{ab|\gamma}$ and $\mathbf{V}_{b|\gamma}$ are diverging for $\mu \rightarrow \infty$. As a result, we can use the formula of Eq. (28) for the asymptotic coherent information, which gives

$$I(a|b) = \log \frac{2}{e} \sqrt{\frac{\det \mathbf{V}_{b|\gamma}}{\det \mathbf{V}_{ab|\gamma}}} = \log \frac{2}{e\varepsilon}. \quad (107)$$

Thus, the PST eigenvalue of Eq. (104) contains all the information about the distribution and distillation of entanglement in the swapping scenario. For $\varepsilon < 1$ entanglement is successfully distributed by the swapping protocol (log-negativity $\mathcal{E} > 0$). Then, for the stronger condition $\varepsilon < 2e^{-1} \simeq 0.73$, the swapped entanglement can also be distilled into $I(a|b)$ entanglement bits per copy by means of one-way protocols. Note that the condition for entanglement distillation for the indirect distribution ($\varepsilon < 2e^{-1}$) is weaker than that found for the direct distribution (equal to $\varepsilon < e^{-1}$).

Now, let us assume the condition of one-mode EB ($\omega = \omega_{EB}$) so that the bipartite states before measurement $\rho_{aA'}$ and $\rho_{B'b}$ are separable (see Fig. 11). We investigate the entanglement in the remote modes a and b by computing the eigenvalue $\varepsilon(\tau, \omega_{EB}, g, g')$. In the standard memoryless case ($\mathbf{G} = \mathbf{0}$) we have $\varepsilon = 1 + \tau^{-1}$ which means that no entanglement can be swapped, as expected. For studying the general correlated environment, we consider different numerical values of the transmissivity τ , and we plot the $\varepsilon(\tau, \omega_{EB}, g, g')$ on the correlation plan. The results are shown in Fig. 12 and are similar to those achieved for direct distribution (see Fig. 5).

In each panel of Fig. 12, the bona-fide values for the correlation parameters are individuated by the non-black area. Remote entanglement is distributed ($\varepsilon < 1$) for values of the correlation parameters (g, g') belonging to the gray activation area. For $\tau = 1/2$, the activation area is populated by entangled environments only. The property that entangled environments are necessary for swapping entanglement is valid for any $\tau \leq 1/2$. In fact, suppose

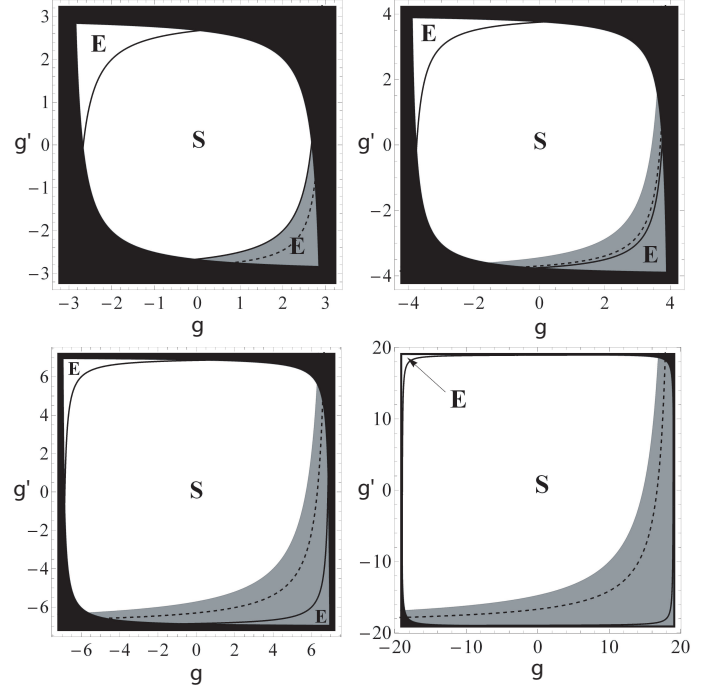


FIG. 12: Analysis of the swapped entanglement ε in the correlation plan for different values of the transmissivity $\tau = 0.5, 0.6, 0.75$, and 0.9 (from top left to bottom right). Corresponding values of the thermal noise are determined by the one-mode EB condition $\omega = \omega_{EB}$. In each inset, the non-black area identifies the set of physical environments, which are divided into separable (S) and entangled (E) environments. The gray region identifies the activation area, i.e., those environments whose correlations are strong enough to generate remote entanglement ($\varepsilon < 1$). Within the gray activation area, the environments that lie below the dashed curve are those able to generate distillable entanglement ($\varepsilon < 2e^{-1}$).

that $\varepsilon < 1$ holds. By using its formula in Eq. (104) and the bona-fide conditions of Eqs. (46) and (48), we can write $\varepsilon^2 < 1$ as

$$\omega^2 - gg' + \omega(g' - g) < \left(\frac{\tau}{1-\tau} \right)^2. \quad (108)$$

Now, for $\tau \leq 1/2$, we have $\tau^2(1-\tau)^{-2} \leq 1$ and using this inequality in Eq. (108), we derive

$$\omega^2 - gg' - 1 < \omega(g - g') \leq \omega|g - g'|, \quad (109)$$

which is the entanglement condition for the environment [i.e., the violation of Eq. (52)].

It is clear that the most interesting result holds for transmissivities $\tau > 1/2$. In this regime, in fact, the distribution of remote entanglement can be activated by separable environments. As explicitly shown for $\tau = 0.6, 0.75$ and 0.9 , the activation area is progressively invaded by separable environments, with the entangled environments almost disappearing for $\tau = 0.9$. In other words, separable correlations become more and more important for increasing transmissivities. Furthermore, for

$\tau \gtrsim 0.75$, separable environments are even able to distribute distillable entanglement ($\varepsilon < 2e^{-1}$).

By comparing Fig. 5 and Fig. 12, e.g., the two corresponding insets for $\tau = 0.75$, it is evident how entanglement is more easily generated by the direct protocol, as a consequence of the extra factor τ^{-1} in Eq. (104). However, it is also evident that distillable entanglement is more easily generated by the swapping protocol, as consequence of the factor 2 in Eq. (107) which leads to the weaker condition $\varepsilon < 2e^{-1}$. This is particularly evident at high transmissivities (where τ^{-1} becomes less important).

It is important to note that our findings suggest a preferred working mechanism for quantum repeaters in the presence of correlated-noise environments, where the swapping protocol is performed before the entanglement distillation stage. In other words, Charlie first performs the swapping protocol on each pair of systems received from Alice and Bob. Correspondingly, the remote systems at Alice's and Bob's stations are locally stored in quantum memories. Finally, these memories are manipulated by the coherent operations of the entanglement distillation protocol. This approach may be successful contrarily to the other strategy, based on distillation followed by swapping, which clearly cannot work in the present scenario.

To conclude, it is easy to specialize the previous analysis of the swapping protocol to the case of SC and AC environments, finding results similar to those found for the direct distribution. For brevity, here we consider only the extremal cases of the MSC and EPR environments (see Fig. 3). As we know the MSC environments are optimal among the separable SC environments. Assuming one-mode EB, the remote entanglement generated by this environment is quantified by

$$\varepsilon_{\text{MSC}} = \frac{1}{\tau} \sqrt{(1-\tau)(1+3\tau)}. \quad (110)$$

One can easily check that $\varepsilon < 1$ if and only if

$$\tau > (1 + \sqrt{5})/4 \approx 0.809. \quad (111)$$

Thus, for sufficiently high transmissivities, the separable correlations of the MSC environment are able to activate the swapping of entanglement, despite the single-mode lossy channels are EB. It is also easy to check that distillable entanglement ($\varepsilon < 2/e$) is generated for

$$\tau > \frac{e(e + 2\sqrt{1+e^2})}{4 + 3e^2} \approx 0.884, \quad (112)$$

which is a regime easily accessible to Alice and Bob.

For the positive EPR environment ($g = |g_{\text{EPR}}|$) we have

$$\varepsilon_{\text{EPR}} = \frac{(1 - \sqrt{\tau})^2}{\tau}, \quad (113)$$

which is less than 1 when $\tau > 1/4$. As expected, remote entanglement can be generated at transmissivities which

are sensibly lower than those necessary for the MSC environments. Furthermore, we can perform entanglement distillation ($\varepsilon < 2/e$) for transmissivities

$$\tau > \frac{e(\sqrt{e} - \sqrt{2})^2}{(e - 2)^2} \simeq 0.289. \quad (114)$$

VI. CONCLUSION AND DISCUSSION

In conclusion, we have investigated the distribution of entanglement in the presence of correlated-noise environments, considering the framework of continuous variable systems, in particular, bosonic systems and Gaussian states. We have introduced and fully characterized a model of correlated Gaussian environment which generalizes the standard model of Gaussian environment with losses and thermal noise.

Considering this correlated-noise environment, we have analyzed scenarios of direct distribution and indirect distribution, i.e., entanglement swapping. In both cases, we have assumed the condition of one-mode EB, meaning that the transmission of a single bosonic mode cannot distribute entanglement between the parties, neither between Charlie and Alice nor between Charlie and Bob. Despite this, we have shown that the distribution of entanglement is still possible when we consider the transmission of two bosonic modes, for instance, from Charlie to Alice and Bob in the scheme of direct distribution.

The success of the two-mode transmission relies on the fact that environmental correlations are injected into the quantum systems and they can be strong enough to restore the entanglement broken by the thermal noise. While this process of reactivation is understandable in the presence of an entangled environment, where the original systems' entanglement is assisted or even replaced by the environmental entanglement, it is clearly paradoxical in the case of a separable environment, where no injection of entanglement may take place.

Surprisingly, the injection of the weaker separable correlations is sufficient to restore the entanglement distribution, as we have shown for wide regimes of parameters. Furthermore, the generated entanglement can even be distillable by means of one-way protocols. The fact that separability can be exploited to recover from entanglement breaking is clearly a paradoxical behavior which poses fundamental questions on the intimate relations between local and nonlocal correlations. In a few words, why do separable correlations have entangling power?

In order to give more analytical examples, we have also defined two subclasses of correlated Gaussian environments, corresponding to symmetric or asymmetric correlations among the quadratures. Using these specific environments, we have analyzed the evolution of the EPR correlations and studied the distribution of entanglement in terms of purely classical correlations and quantum discord. We have found that entanglement can be restored by separable correlations which are mostly classical, i.e.,

with a negligible amount of quantum discord.

Besides providing the first treatment of the problem of entanglement breaking in the presence of correlated noise, our work also extends the theory of entanglement swapping to this scenario, suggesting a preferred mechanism for quantum repeaters, where the swapping protocol must anticipate the distillation stage.

In terms of potential impact, our analysis shows new perspectives for entanglement distribution and distillation in the presence of memory channels and correlated-noise environments, such as those arising from the non-Markovian dynamics of open quantum systems [50, 51]. Note that memory channels and non-Markovian environments are present in a wide series of practical scenarios. For instance, they naturally arise in the context of spin chains [20] and micromasers [21]. Other important examples can be found in condensed matter, in particular when we consider the dynamics of quantum dots in photonic crystals [22]. In the bosonic setting, memory Gaussian channels come into play when electromagnetic modes propagate through dispersive media, such as linear optical systems or free-space. In this case, correlations and memory effects are naturally introduced by diffraction [23–25]. Finally, other examples of bosonic memory channels can be found in the radiation propagation through atmospheric turbulence [26–28].

Appendix A: Computation of the covariance matrix of Eq. (102)

Consider the scenario depicted in Fig. 11. We have a total of 6 input modes: Alice's modes a and A , Bob's modes b and B , and Eve's modes E_1 and E_2 . The global input state is in a tensor product

$$\rho_{aA} \otimes \rho_{bB} \otimes \rho_{E_1 E_2}, \quad (\text{A1})$$

where $\rho_{aA} = \rho_{bB} = \rho$ is an input EPR state with CM given in Eq. (29) and $\rho_{E_1 E_2}$ is an environmental zero-mean Gaussian state with CM given in Eq. (42). The global input state is therefore a zero-mean Gaussian state with CM

$$\mathbf{V}_{aAbBE_1E_2} = \mathbf{V}(\mu) \oplus \mathbf{V}(\mu) \oplus \mathbf{V}_{E_1E_2}(\omega, g, g'). \quad (\text{A2})$$

It is helpful to permute the modes so to have the ordering $abAE_1E_2B$, where the upper-case modes are those transformed by the beam splitters. After reordering, the input CM has the explicit form

$$\mathbf{V}_{abAE_1E_2B} = \begin{pmatrix} \mu\mathbf{I} & \mathbf{0} & \mu'\mathbf{Z} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mu\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mu'\mathbf{Z} \\ \mu'\mathbf{Z} & \mathbf{0} & \mu\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \omega\mathbf{I} & \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G} & \omega\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mu'\mathbf{Z} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mu\mathbf{I} \end{pmatrix}, \quad (\text{A3})$$

where $\mathbf{0}$ is the 2×2 zero matrix. Now the global action of the two beam splitters can be represented by the

symplectic matrix

$$\mathbf{S} = \mathbf{I} \oplus \mathbf{I} \oplus \mathbf{S}(\tau) \oplus \mathbf{S}(\tau)^T, \quad (\text{A4})$$

where the identity matrices $\mathbf{I} \oplus \mathbf{I}$ act on the remote modes, a and b , the beam splitter matrix $\mathbf{S}(\tau)$ [given in Eq. (11)] acts on modes A and E_1 , while $\mathbf{S}(\tau)^T$ acts on modes E_2 and B . The second beam-splitter matrix is transposed in order to have positive reflection from the environment.

The output state of modes $abA'E'_1E'_2B'$ after the action of the interferometer is a Gaussian state with zero mean and CM equal to

$$\mathbf{V}_{abA'E'_1E'_2B'} = \mathbf{S} \mathbf{V}_{abAE_1E_2B} \mathbf{S}^T. \quad (\text{A5})$$

After simple algebra, we get

$$\mathbf{V}_{abA'E'_1E'_2B'} = \begin{pmatrix} \mathbf{V}_{ab} & \mathbf{W}_1 & \mathbf{W}_2 \\ \mathbf{W}_1^T & \mathbf{V}_{A'E'_1} & \mathbf{W}_3 \\ \mathbf{W}_2^T & \mathbf{W}_3^T & \mathbf{V}_{E'_2B'} \end{pmatrix}, \quad (\text{A6})$$

where the blocks along the diagonal correspond to the reduced CMs $\mathbf{V}_{ab} = \mu(\mathbf{I} \oplus \mathbf{I})$,

$$\mathbf{V}_{A'E'_1} = \begin{pmatrix} x\mathbf{I} & z\mathbf{I} \\ z\mathbf{I} & x'\mathbf{I} \end{pmatrix}, \quad \mathbf{V}_{E'_2B'} = \begin{pmatrix} x'\mathbf{I} & z\mathbf{I} \\ z\mathbf{I} & x\mathbf{I} \end{pmatrix}, \quad (\text{A7})$$

with

$$x := \tau\mu + (1-\tau)\omega, \quad x' := \tau\omega + (1-\tau)\mu, \quad (\text{A8})$$

and $z := \sqrt{\tau(1-\tau)}(\omega - \mu)$. The off-diagonal blocks are given by

$$\mathbf{W}_1 = \begin{pmatrix} \mu'\sqrt{\tau}\mathbf{Z} & -\mu'\sqrt{1-\tau}\mathbf{Z} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad (\text{A9})$$

$$\mathbf{W}_2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -\mu'\sqrt{1-\tau}\mathbf{Z} & \mu'\sqrt{\tau}\mathbf{Z} \end{pmatrix}, \quad (\text{A10})$$

and

$$\mathbf{W}_3 = \begin{pmatrix} \sqrt{\tau(1-\tau)}\mathbf{G} & (1-\tau)\mathbf{G} \\ \tau\mathbf{G} & \sqrt{\tau(1-\tau)}\mathbf{G} \end{pmatrix}. \quad (\text{A11})$$

Since we are interested in the CM of Alice and Bob, we trace out the two environmental modes E'_1 and E'_2 , which corresponds to deleting the corresponding rows and columns in the CM of Eq. (A6). As a result, we get the following reduced CM for the modes $abA'B'$

$$\mathbf{V}_{abA'B'} = \begin{pmatrix} \mu\mathbf{I} & \mathbf{0} & \mu'\sqrt{\tau}\mathbf{Z} & \mathbf{0} \\ \mathbf{0} & \mu\mathbf{I} & \mathbf{0} & \mu'\sqrt{\tau}\mathbf{Z} \\ \mu'\sqrt{\tau}\mathbf{Z} & \mathbf{0} & x\mathbf{I} & (1-\tau)\mathbf{G} \\ \mathbf{0} & \mu'\sqrt{\tau}\mathbf{Z} & (1-\tau)\mathbf{G} & x\mathbf{I} \end{pmatrix}. \quad (\text{A12})$$

From this matrix, it is easy to check that the bipartite states before measurement $\rho_{aA'}$ and $\rho_{bB'}$ have CMs $\mathbf{V}_{aA'} = \mathbf{V}_{bB'}$ of the same form given in Eq. (64). It is also trivial to check that Alice and Bob's state $\rho_{A'B'}$ is

asymptotically separable for any $\tau > 0$. In fact, from the reduced CM $\mathbf{V}_{A'B'}$, we can compute the smallest PTS eigenvalue

$$\varepsilon_{A'B'} = \tau\mu + \frac{1-\tau}{2}(2\omega - |g - g'|), \quad (\text{A13})$$

which is always > 1 for large μ and $\tau > 0$. In the singular case $\tau = 0$, we have $\rho_{A'B'} = \rho_{E_1 E_2}$ so that entanglement or separability directly comes from the environment.

Most importantly, from the CM of Eq. (A12) we can compute the CM $\mathbf{V}_{ab|\gamma}$ of the conditional remote state $\rho_{ab|\gamma}$ after the Bell measurement. In order to achieve this result, we use the transformation formula for CMs under Bell measurements which has been proven in Ref. [52]. As a first step, we put $\mathbf{V}_{abA'B'}$ in the blockform

$$\mathbf{V}_{abA'B'} = \begin{pmatrix} \mathbf{A} & \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{C}_1^T & \mathbf{B}_1 & \mathbf{D} \\ \mathbf{C}_2^T & \mathbf{D}^T & \mathbf{B}_2 \end{pmatrix}, \quad (\text{A14})$$

where

$$\mathbf{A} = \mu\mathbf{I} \oplus \mu\mathbf{I}, \quad \mathbf{B}_1 = \mathbf{B}_2 = x\mathbf{I}, \quad \mathbf{D} = (1 - \tau)\mathbf{G}, \quad (\text{A15})$$

and

$$\mathbf{C}_1 = \begin{pmatrix} \mu'\sqrt{\tau}\mathbf{Z} \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{C}_2 = \begin{pmatrix} \mathbf{0} \\ \mu'\sqrt{\tau}\mathbf{Z} \end{pmatrix}. \quad (\text{A16})$$

From the blocks of the CM (A14), we construct the following matrix

$$\boldsymbol{\Theta} = \frac{1}{2}(\mathbf{Z}\mathbf{B}_1\mathbf{Z} + \mathbf{B}_2 - \mathbf{Z}\mathbf{D} - \mathbf{D}^T\mathbf{Z}), \quad (\text{A17})$$

where \mathbf{Z} is the reflection matrix of Eq. (30). Then the conditional CM is given by the formula [52]

$$\mathbf{V}_{ab|\gamma} = \mathbf{A} - \frac{1}{2\det\boldsymbol{\Theta}} \sum_{i,j=1}^2 \mathbf{C}_i(\mathbf{X}_i^T \boldsymbol{\Theta} \mathbf{X}_j) \mathbf{C}_j^T, \quad (\text{A18})$$

where

$$\mathbf{X}_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \mathbf{X}_2 := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \boldsymbol{\Omega}. \quad (\text{A19})$$

Explicitly, we compute the theta matrix

$$\boldsymbol{\Theta} = \begin{pmatrix} \theta & 0 \\ 0 & \theta' \end{pmatrix}, \quad (\text{A20})$$

with elements given in Eq. (103). Using this matrix in Eq. (A18) we get the final expression of Eq. (102). It is easy to check that for $\tau = 1$ (absence of loss and noise), we get Eq. (94) which is the result already known in the literature [49, 53].

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